Bernoulli Convolutions Associated with some Algebraic Numbers

KWOK, Tsz Chiu

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Thesis/Assessment Committee

Professor LEUNG Chi Wai (Chair)

Professor FENG Dejun (Thesis Supervisor)

Professor NG Kung Fu (Committee Member)

Professor Nikita SIDOROV (External Examiner)

Abstract

Bernoulli convolutions have been studied since 1930's, revealing strong connections with other areas in mathematics such as harmonic analysis, fractal geometry, dynamical systems and number theory. It is well-known that the Bernoulli convolution ν_λ is either purely singular or absolutely continuous, and one of the fundamental questions on Bernoulli convolutions is to determine for which values of $\lambda \in (\frac{1}{2}, 1)$, ν_λ is purely singular. Erdős [3] showed that ν_λ is purely singular when $\lambda \in (\frac{1}{2}, 1)$ is the reciprocal of a PV number. On the other hand, Solomyak [22] proved that ν_λ is absolutely continuous for a.e. $\lambda \in (\frac{1}{2}, 1)$. It is still an open question whether the only purely singular Bernoulli convolutions are those associated with the reciprocal of Pisot numbers.

In this thesis, we give some basic properties of Bernoulli convolutions ν_λ. We present some known results on the singularity of ν_λ, the entropy and dimensions of ν_λ. We also give some examples for which the asymptotical behavier of ν_λ is known.

摘要

Bernoulli 卷積的研究起始自1930年代。它與數學的不同範疇如調和分析、分形幾何、動力系統與數論均有密切的聯繫。 Bernoulli 卷積 ν_λ 有一個著名的特性：ν_λ 只能是一個奇異測度或一個絶對連續測度。因此研究 Bernoulli 卷積一個基本的問題是對於怎樣的 $\lambda \in (\frac{1}{2}, 1)$，$\nu_\lambda$是一個奇異測度。 Erdős 指出當 $\lambda \in (\frac{1}{2}, 1)$ 是一個 PV 數的倒數，則 ν_λ 是一個奇異測度。 另一方面，Solomyak 證明了對於幾乎所有 $\lambda \in (\frac{1}{2}, 1)$，$\nu_\lambda$ 都是絶對連續的。 奇異的 Bernoulli 卷積是否僅發生在 PV 數的倒數，至今仍是懸題。

我們在這篇論文中會討論 Bernoulli 卷積的特性。我們會展示一些已知結果，包括 ν_λ 是奇異的一些條件，ν_λ 的熵和維數，以及討論一些例子。

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KWOK, Tsz Chiu

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Contents

1 **Introduction** 6

1.1 Historical remarks and main results 6

1.2 Structure of the thesis 8

2 **Basic properties** 10

2.1 Existence of infinite convolution 10

2.2 Properties 16

2.3 Law of pure type 17

3 **Some results related to pure singularity** 20

3.1 The Pisot-Vijayaraghavan numbers 20

3.2 The Salem numbers 22

3.3 The weak separation condition 23

4 **A proof of almost everywhere absolute continuity** 30

5 **Other results and problems** 37

5.1 Entropy of Bernoulli convolutions 37

5.2 Dimensions 40

5.3 Non PV numbers with bad behavior 41

5.4 Open problems 41

Bibliography **43**

Chapter 1

Introduction

For $\lambda \in (0,1)$, let ν_λ be the probability distribution of the random variable $X = \sum_{i=0}^{\infty} \pm\lambda^i$, where the signs are chosen independently with probabilities $(\frac{1}{2}, \frac{1}{2})$. This family of measures are called Bernoulli convolutions.

Bernoulli convolutions have surprising connections with a number of other fields in mathematics, including harmonic analysis, fractal geometry, dynamical systems, number theory and estimation of Hausdorff dimension. In this thesis, we will discuss some properties and results related to the Bernoulli convolutions.

In the following sections, we will first give some historical remarks on the Bernoulli convolutions, and then explains the structure of this thesis.

1.1 Historical remarks and main results

Bernoulli convolutions have been studied since the 1930's. Jessen and Wintner [10] proved that ν_λ is either purely singular or absolutely continuous with respect to the Lebesgue measure.

Then one of the fundamental problem on Bernoulli convolutions is that for which $\lambda \in (0,1)$, ν_λ is purely singular. For $\lambda \in (0, \frac{1}{2})$, Kershner and Wintner [12] proved that ν_λ is purely singular since it is supported in a standard Cantor set which has zero Lebesgue measure. Actually it is the standard Cantor measure. Wintner [23] noticed that for $\lambda = \frac{1}{2}$, ν_λ is uniform in its support $[-2, 2]$, and for $\lambda = 2^{-1/k}$ with $k \geq 2$, ν_λ is absolutely continuous with a density in $C^{k-2}(\mathbb{R})$. For $\lambda \in (\frac{1}{2}, 1)$, the support of ν_λ is the interval $[-(1-\lambda)^{-1}, (1-\lambda)^{-1}]$.

It is then natural to think that ν_λ is absolutely continuous for all such λ. However Erdős [3] proved ν_λ is purely singular when λ is the reciprocal of a Pisot-Vijayaraghavan number, by showing that the Fourier transform of the Bernoulli convolutions $\widehat{\nu_\lambda}$ does not tend to zero at infinity. A PV number is a real algebraic integer greater than 1, and all its conjugates are of magnitude strictly less than 1. An example of PV numbers is the golden ratio $\frac{1+\sqrt{5}}{2}$. One year later, Erdős [4] proved a result in the opposite direction, namely there exists $\gamma < 1$ such that for almost all $\lambda \in (\gamma, 1)$, ν_λ is absolutely continuous.

Up to now the reciprocals of PV numbers are the only known examples where $\lambda \in (\frac{1}{2}, 1)$ but ν_λ is purely singular. Also the method used by Erdős cannot generate any more examples, because Salem [19] showed that for all $\lambda \in (\frac{1}{2}, 1)$ such that λ^{-1} is not a PV number, $\widehat{\nu_\lambda}$ tends to zero at infinity.

On the other hand, Garsia [6] found the largest known explicit set of λ where ν_λ is absolutely continuous with bounded density. The set consists of reciprocals of algebraic integers in $(1, 2)$ whose minimal polynomial has constant coefficient ± 2 and has other roots outside unit disk. For instance, the real root of $x^{n+p} - x^n - 2 = 0$ where $p, n \geq 1$ and $\max\{p, n\} \geq 2$ satisfies the criterion.

In 1995, Solomyak [22] proved a fundamental theorem which states that for almost all $\lambda \in (\frac{1}{2}, 1)$, ν_λ is absolutely continuous with L^2 density. Soon afterwards, Peres and Solomyak [21] gave a considerably simpler proof.

Since Solomyak showed that for almost all $\lambda \in (\frac{1}{2}, 1)$, ν_λ has a density function in $L^2(\mathbb{R})$, it is then natural to ask for values of λ so that ν_λ does not have L^2 density. In 2004, Feng and Wang [5] gave a positive answer to this question by finding a set of algebraic integers which are non PV numbers and do not have L^2

density.

Apart from finding for which values of λ, ν_λ is purely singular, another interesting question related to Bernoulli convolutions is to estimate its dimensions. Garsia [7] introduced a new notion associated with Bernoulli convolutions with $\lambda \in (\frac{1}{2}, 1)$. Let

$$d_N(\lambda) = \left\{ x \in \mathbb{R} \,:\, x = \sum_{n=0}^{N-1} a_n \lambda^n, a_n \in \{-1, 1\} \right\},$$

and for each $x \in d_N(\lambda)$, let

$$p_N(x) = 2^{-N} \# \left\{ (a_0, a_1, \ldots, a_{N-1}) \in \{-1, 1\}^N \,:\, x = \sum_{n=0}^{N-1} a_n \lambda^n \right\}.$$

Finally,

$$H_N(\lambda) = -\sum_{x \in d_N(\lambda)} p_n(x) \log p_n(x) \quad \text{and} \quad G_\lambda = \lim_{N \to \infty} \frac{H_N(\lambda)}{N}. \tag{1.1}$$

In [7] Garsia proved the limit always exists when $\lambda \in (\frac{1}{2}, 1)$, and if λ^{-1} is a PV number, then $G_\lambda < -\log \lambda$. Also, if $G_\lambda < -\log \lambda$, then ν_λ is purely singular. However, so far no non PV numbers have been found to satisfy this property. $H_\lambda = \frac{G_\lambda}{-\log \lambda}$ is called *Garsia's entropy.* Later, Lalley [13] proved that when λ is the reciprocal of a PV number, the Garsia's entropy coincides with the Hausdorff dimension of ν_λ.

Though Equation 1.1 looks simple, it is hard to approximate H_λ using this equation directly. For $\lambda = \lambda_g = \frac{\sqrt{5}-1}{2}$, Alexander and Zagier [1] managed to evaluate H_λ with high precision by considering the Fibonacci graph. It turns out that $\dim \nu_{\lambda_g} = H_{\lambda_g} \approx 0.9957$. Hare and Sidorov [8] showed that $H_\lambda > 0.81$ for all $\lambda \in (\frac{1}{2}, 1)$ and λ^{-1} is a PV number.

1.2 Structure of the thesis

In chapter 2, we give some basic properties related to Bernoulli convolutions. We will explain that ν_λ is well-defined, show some of its identifications, and prove that it is either purely singular or absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$.

In chapter 3, we study some results related to the pure singularity of ν_λ. We will prove that ν_λ is purely singular when λ is the reciprocal of a PV number following the idea of Erdős. Also we will introduce the Salem numbers and explain the reason the reciprocals of Salem numbers are believed to be good candidates where the Bernoulli convolutions are purely singular. Finally we will introduce the weak separation condition which provides a way to check singularity of Bernoulli convolutions.

In chapter 4, we present the result of Peres and Solomyak [21] which shows ν_λ is absolutely continuous for almost all $\lambda \in (\frac{1}{2}, 1)$. The proof relies on the transversality property.

In chapter 5, we give some other results and some open questions related to Bernoulli convolutions. We will present the method of Hare and Sidorov [8] which gives a global lower bound the Garsia's entropy, and show some results related to estimation of entropy and dimensions. Next we will state some current results on smoothness of ν_λ and conclude our thesis with some open problems.

Chapter 2

Basic properties

In this chapter, we aim at explaining several basic properties of Bernoulli convolutions. Firstly we will show that for $\lambda \in (0,1)$, ν_λ is well-defined and satisfies

$$\nu_\lambda(\cdot) = \frac{1}{2}\left(\nu_\lambda\left(\frac{\cdot+1}{\lambda}\right) + \nu_\lambda\left(\frac{\cdot-1}{\lambda}\right)\right).$$

The method we use depends on self similarity of Bernoulli convolutions. Then we find the Fourier transform of ν_λ and show that ν_λ can be viewed as a "non-linear projection" of sequence space. Finally we will prove the law of pure type, that is, ν_λ is either purely singular or absolutely continuous with respect to $\mathcal{L}^1$.

2.1 Existence of infinite convolution

For $x \in \mathbb{R}$, we let δ_x denote the Dirac measure at x, that is, for any Borel set $A \subseteq \mathbb{R}$,

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{otherwise.} \end{cases}$$

The measure $b_x := \frac{1}{2}(\delta_{-x} + \delta_x)$ is called a Bernoulli measure.

We use $\mu * \nu$ to denote the convolution of two measures μ and ν. The convolution is defined by for any Borel set $A \subseteq \mathbb{R}$,

$$(\mu * \nu)(A) := \int_{\mathbb{R}} \mu(A - x) d\nu(x).$$

If μ and ν are probability measures, then the convolution $\mu * \nu$ is the probability distribution of the sum $X + Y$ of two independent random variables X and Y whose respective distributions are μ and ν.

Let $BC(\mathbb{R})$ be the set of bounded and continuous functions from $\mathbb{R}$ to $\mathbb{R}$. We say that a sequence of measures $\{\mu_n\}_{n=1}^{\infty}$ converges weakly to a measure μ if for any $f \in BC(\mathbb{R})$,

$$\mu_n(f) := \int_{\mathbb{R}} f d\mu_n \to \mu(f).$$

For $0 < \lambda < 1$, it can be shown that the measures

$$\nu_\lambda^n = b_1 * b_\lambda * b_{\lambda^2} * \ldots * b_{\lambda^{n-1}}.$$

converge weakly to a measure. This limit is denoted by ν_λ and is the infinite convolution of Bernoulli measures associated with the parameter $0 < \lambda < 1$. It is often called *infinitely convolved Bernoulli measure*(ICBM), or simply *Bernoulli convolution.*

Now we would like to show that for any $\lambda \in (0, 1)$ the Bernoulli convolution is well-defined, that is, the sequence of partial convolutions really converges weakly. We will make use of the following theorem.

Theorem 2.1. *(Hutchinson [9]) Let $S_i : \mathbb{R} \to \mathbb{R}$ be contractions, $p_i \in [0, 1]$ and $\sum_{i=1}^{N} p_i = 1$. There exists a unique probability Borel measure ν with compact support that satisfies*

$$\nu = \sum_{i=1}^{N} p_i \nu \circ S_i^{-1}.$$

Before the proof of the theorem, let us state some definitions and notations.

Definition 2.2. Let X be a metric space. The *Lipschitz constant* of a function $f : X \to X$ is

$$\operatorname{Lip} f = \sup_{x \neq y \in X} \frac{d(f(x), f(y))}{d(x, y)}.$$

We say f is *Lipschitz* if Lip $f < \infty$. We say f is a *contraction* if Lip $f < 1$.

Definition 2.3. For a Borel regular measure μ, the *support* of μ is defined as

$$\text{spt } \mu = \mathbb{R}\setminus \bigcup\{V \ : \ V \text{ open}, \mu(V) = 0\}.$$

It can be shown that $\mu(\text{spt } \mu) = \mu(\mathbb{R})$.

Notation 2.4. For a non-empty compact interval $I \subseteq \mathbb{R}$, let $M(I)$ denote the collection of all Borel regular probability measures ν on I.

Definition 2.5. For $\mu, \nu \in M(I)$, let

$$L(\mu, \nu) = \sup\{|\mu(\phi) - \nu(\phi)| \ : \ \phi : \mathbb{R} \to \mathbb{R}, \text{Lip } \phi \leq 1\},$$

where $\mu(\phi) := \int_{\mathbb{R}} \phi d\mu$. It will be shown later that L defines a complete metric over $M(I)$.

Definition 2.6. An *iterated function system*(IFS) $\mathcal{S}$ is a finite set of contractions $\{S_i\}_{i=1}^N$. Sometimes an IFS is associated with a probability vector $\{p_i\}_{i=1}^N$ with $p_i \in [0,1]$ and $\sum_{i=1}^N p_i = 1$. The IFS associated with probability vector p is denoted by $(\mathcal{S}, p)$.

Notation 2.7. Given an iterated function system $\mathcal{S}$. For any Borel set $A \subseteq \mathbb{R}$, let

$$S(A) = \bigcup_{i=1}^N S_i(A).$$

For any Borel regular measure μ, let

$$(S,p)(\mu) = \sum_{i=1}^N p_i \mu \circ S_i^{-1}.$$

Usually the probability vector p is understood clearly. In that case we will also use $S(\mu)$ to denote $(S,p)(\mu)$. A Borel set $A \subseteq \mathbb{R}$ is called an *invariant set* under S if $S(A) = A$. Similarly a Borel regular measure μ is called an *invariant measure* under (S,p) (or simply S if p is understood) if $(S,p)(\mu) = \mu$. Notice that if $I \supseteq S(I) \neq \emptyset$ and $\mu \in M(I)$, then $S(\mu) \in M(I)$. It will be shown that (S,p) is a contraction on L metric.

Proof of Theorem 2.1. We will show that L defines a complete metric over $M(I)$ where $I \supseteq S(I) \neq \emptyset$, and $S = (S, p)$ is a contraction on L. Then the result follows from Banach fixed point theorem.

It is obvious that for $\mu, \nu \in M(I)$, $L(\mu,\nu) \geq 0$, $L(\mu,\nu) = 0$ if and only if $\mu = \nu$, and $L(\mu,\nu) = L(\nu,\mu)$. For any ϕ with Lip $\phi \leq 1$,

$$\begin{aligned}|\mu_1(\phi) - \mu_2(\phi)| &\leq |\mu_1(\phi) - \mu_3(\phi)| + |\mu_3(\phi) - \mu_2(\phi)| \\ &\leq L(\mu_1, \mu_3) + L(\mu_3, \mu_2).\end{aligned}$$

The triangle inequality follows by taking supremum over all ϕ with Lip $\phi \leq 1$ on both sides. Finally, suppose $I \subseteq [-R, R]$. Then

$$\begin{aligned}\mu(\phi) - \nu(\phi) &= \mu(\phi - \phi(0)) - \nu(\phi - \phi(0)) + \mu(\phi(0)) - \nu(\phi(0)) \\ &= \mu(\phi - \phi(0)) - \nu(\phi - \phi(0)) \\ &\leq \mu(R) + \nu(R) \\ &= 2R,\end{aligned}$$

where the second equality holds because $\mu(c) = \nu(c) = c$ for any constant c. Therefore $L(\mu,\nu) \leq 2R < \infty$ for any $\mu, \nu \in M(I)$ and L is indeed a metric.

Suppose $\{\mu_n \in M(I)\}$ is Cauchy in L. For any Lipschitz ϕ, let $r = \text{Lip } \phi$. We have

$$\begin{aligned}|\mu_n(\phi) - \mu_m(\phi)| &= r|\mu_n(r^{-1}\phi) - \mu_m(r^{-1}\phi)| \\ &\leq rL(\mu_n, \mu_m) \quad \text{since Lip } (r^{-1}\phi) = 1 \\ &\to 0, \quad \text{as } \min(m,n) \to \infty.\end{aligned}$$

Now for any $\phi \in C_c(\mathbb{R})$, for any $\epsilon > 0$, there exists a Lipschitz ψ such that $\sup_{x \in I} |\phi(x) - \psi(x)| < \epsilon$. So

$$\begin{aligned}|\mu_n(\phi) - \mu_m(\phi)| &\leq |\mu_n(\phi) - \mu_n(\psi)| + |\mu_n(\psi) - \mu_m(\psi)| + |\mu_m(\psi) - \mu_m(\phi)| \\ &= |\mu_n(\phi - \psi)| + |\mu_n(\psi) - \mu_m(\psi)| + |\mu_m(\psi - \phi)| \\ &\leq 2\epsilon + |\mu_n(\psi) - \mu_m(\psi)| \to 2\epsilon.\end{aligned}$$

Therefore $\{\mu_n(\phi)\}$ is a Cauchy sequence for any $\phi \in C_c(\mathbb{R})$, and $\Lambda\phi = \lim_{n\to\infty} \mu_n(\phi)$ is well-defined.

Obviously Λ is a positive linear functional on $C_c(\mathbb{R})$. By Riesz representation theorem, there exists a Borel regular measure μ such that $\Lambda\phi = \mu(\phi)$ for any $\phi \in C_c(\mathbb{R})$. Clearly spt $\mu \in I$ and $\mu(\mathbb{R}) = \mu(\chi_I) = \lim_{n\to\infty} \mu_n(\chi_I) = 1$, therefore $\mu \in M(I)$. Finally for any given $\epsilon > 0$, there exists N such that for any $n, m > N$ and any ϕ satisfying Lip $\phi \leq 1$, $\mu_n(\phi) - \mu_m(\phi)| \leq \epsilon$. This implies for any $\epsilon > 0$, there exists N such that for any $n > N$ and any ϕ satisfying Lip $\phi \leq 1$, $|\mu_n(\phi) - \mu(\phi)| \leq \epsilon$. Therefore L is complete.

Now we have to show that S is a contraction on $M(I)$. Notice that if $\mu' = \mu \circ S_i^{-1}$, then $\mu'(f) = \mu(f \circ S_i)$. So $S(\mu)(f) = \sum_{i=1}^{N} p_i\mu(f \circ S_i)$. Let $r = \max_{1\leq i\leq N} \text{Lip } S_i < 1$.

For any ϕ with Lip $\phi \leq 1$,

$$\begin{aligned}
|S(\mu)(\phi) - S(\nu)(\phi)| &= |\sum_{i=1}^{N} p_i\mu(\phi \circ S_i) - \nu(\phi \circ S_i)| \\
&\leq \sum_{i=1}^{N} p_i|\mu(\phi \circ S_i) - \nu(\phi \circ S_i)| \\
&= r\sum_{i=1}^{N} p_i|\mu(r^{-1}\phi \circ S_i) - \nu(r^{-1}\phi \circ S_i)| \\
&\leq r\sum_{i=1}^{N} p_i L(\mu, \nu) \\
&= rL(\mu, \nu).
\end{aligned}$$

The second last inequality holds since Lip $r^{-1}\phi \circ S_i \leq r^{-1}(\text{Lip } \phi)(\text{Lip } S_i) \leq 1$. Taking supremum over all ϕ with Lip $\phi \leq 1$, we have $L(S(\mu), S(\nu)) \leq rL(\mu, \nu)$. As $r < 1$, S is a contraction on L.

Now take I to be a compact interval satisfying $I \supseteq S(I) \neq \emptyset$. By Banach fixed point theorem, there exists a unique member $\nu \in M(I)$ satisfying $\nu = \sum_{i=1}^{N} p_i\nu \circ S_i^{-1}$. So the existence of the required measure in the theorem is guaranteed.

Suppose μ_1, μ_2 are both probability Borel regular measures with compact support that satisfies $\nu = S(\nu)$, take $I \supseteq \text{spt } \mu_1 \bigcup \text{spt } \mu_2$. Then $\mu_i \in M(I)$. By uniqueness of fixed point, $\mu_1 = \mu_2$. $\square$

Corollary 2.8. *Let $\nu_\lambda^n = b_1 * b_\lambda * b_{\lambda^2} * \ldots * b_{\lambda^{n-1}}$. ν_λ^n converges to ν_λ weakly, where ν_λ is the unique probability measure with compact support that satisfies*

$$\nu(\cdot) = \frac{1}{2}\left(\nu\left(\frac{\cdot + 1}{\lambda}\right) + \nu\left(\frac{\cdot - 1}{\lambda}\right)\right).$$

Proof. Let $S_1(x) = \lambda x + 1, S_2(x) = \lambda x - 1$, then S_i are contractions. From the proof of the theorem, we can see that $S^{(n)}(b_1)$ converge to the unique probability measure with compact support ν_λ that satisfies $S(\nu_\lambda) = \nu_\lambda$ in the sense of L metric. It remains to show that $S(\nu_\lambda^n) = \nu_\lambda^{n+1}$ and convergence in the sense of L metric implies weak convergence.

Notice that for any $n \geq 1$,

$$\nu_\lambda^n = 2^{-n} \sum_{\{a_i\}_0^{n-1} \in \{-1,1\}^n} \delta_{\sum_{i=0}^{n-1} a_i \lambda^i},$$

and that

$$S(\delta_x) = \frac{1}{2}(\delta_{\lambda x+1} + \delta_{\lambda x-1}).$$

Therefore

$$\begin{aligned}
\nu_\lambda^{n+1} &= 2^{-n-1} \sum_{\{a_i\}_0^n \in \{-1,1\}^{n+1}} \delta_{\sum_{i=0}^n a_i \lambda^i} \\
&= 2^{-n-1} \sum_{\{a_i\}_1^n \in \{-1,1\}^n} (\delta_{\lambda \sum_{i=1}^n a_i \lambda^{i-1}+1} + \delta_{\lambda \sum_{i=1}^n a_i \lambda^{i-1}-1}) \\
&= \frac{1}{2} \cdot 2^{-n} \sum_{\{a_i\}_0^{n-1} \in \{-1,1\}^n} (\delta_{\lambda \sum_{i=0}^{n-1} a_i \lambda^i+1} + \delta_{\lambda \sum_{i=0}^{n-1} a_i \lambda^i-1}) \quad \text{by reindexing} \\
&= S(\nu_\lambda^n).
\end{aligned}$$

Finally by the construction of Cauchy limit, μ_n converges to μ in the sense of L metric implies $\mu_n(\phi)$ converges to $\mu(\phi)$ for any $\phi \in C_c(\mathbb{R})$. Since μ_n and μ have support in compact interval I, the same applies for any function in $BC(\mathbb{R})$ by considering a function in $C_c(\mathbb{R})$ that coincide in I. Therefore ν_λ^n converges to ν_λ weakly. $\square$

2.2 Properties

Apart from being the unique invariant probability measure with respect to S, ν_λ can be viewed in some other useful ways. Some of its properties can be revealed by considering its Fourier transform, which can be easily computed.

Definition 2.9. The Fourier transform of a finite Borel measure μ on $\mathbb{R}$ is a function $\widehat{\mu} : \mathbb{R} \to \mathbb{R}$ given by

$$\widehat{\mu}(\xi) = \mu(e^{-it\xi}) = \int_{-\infty}^{\infty} e^{-it\xi} d\mu(t).$$

Recall that Fourier transform and convolutions have a close relation, namely for any finite Borel measures μ and ν on $\mathbb{R}$,

$$\widehat{\mu * \nu} = \widehat{\mu} \cdot \widehat{\nu}. \tag{2.1}$$

Proposition 2.10. *The Fourier transform of* ν_λ *is*

$$\widehat{\nu_\lambda}(\xi) = \prod_{n=0}^{\infty} \cos(\lambda^n \xi).$$

Proof. First,

$$\widehat{\delta_x}(\xi) = \int_{-\infty}^{\infty} e^{-it\xi} d\delta_x(t) = e^{-ix\xi}.$$

Therefore,

$$\widehat{b_x}(\xi) = \frac{1}{2}(\widehat{\delta_x} + \widehat{\delta_{-x}}) = \frac{1}{2}(e^{-ix\xi} + e^{ix\xi}) = \cos(x\xi).$$

By (2.1),

$$\widehat{\nu_\lambda^n} = \prod_{m=0}^{n-1} \widehat{b_{\lambda^m}}(\xi) = \prod_{m=0}^{n-1} \cos(\lambda^m \xi).$$

Since the function $e^{-it\xi} \in BC(\mathbb{R})$,

$$\widehat{\nu_\lambda}(\xi) = \lim_{n\to\infty} \widehat{\nu_\lambda^n}(\xi) = \prod_{m=0}^{\infty} \cos(\lambda^m \xi).$$

□

ν_λ can be viewed as a "non-linear projection" of sequence space.

Proposition 2.11. *Let $X = \{-1, 1\}$ be a measure space equipped with discrete topology and measure P defined by $P(\{-1\}) = P(\{1\}) = \frac{1}{2}$. Let $\Omega = \{\{\omega_i\}_0^\infty \in X^{\mathbb{N}}\}$ be the sequence space associated with the product topology and the product measure μ. Then*

$$\nu_\lambda = \mu \circ \Pi_\lambda^{-1} \quad \textit{where} \quad \Pi_\lambda(\omega) = \sum_{n=0}^{\infty} \omega_n \lambda^n.$$

Proof. Obviously $\mu \circ \Pi_\lambda^{-1}$ is a probability Borel measure on $\mathbb{R}$ with compact support. It remains to show that it is invariant under S.

$$\begin{aligned}
\mu \circ \Pi_\lambda^{-1}(A) &= \mu(\{\omega \, : \, \Pi_\lambda(\omega) \in A\}) \\
&= \mu(\{\omega \, : \, \omega_0 = -1, \Pi_\lambda(\omega) \in A\}) + \mu(\{\omega \, : \, \omega_0 = 1, \Pi_\lambda(\omega) \in A\}) \\
&= \mu(\{(1)\omega \, : \, \Pi_\lambda(\omega) \in S_1^{-1}(A)\}) + \mu(\{(-1)\omega \, : \, \Pi_\lambda(\omega) \in S_2^{-1}(A)\}) \\
&= \frac{1}{2}\mu(\{\omega \, : \, \Pi_\lambda(\omega) \in S_1^{-1}(A)\}) + \frac{1}{2}\mu(\{\omega \, : \, \Pi_\lambda(\omega) \in S_2^{-1}(A)\}) \\
&= \frac{1}{2}(\mu \circ \Pi_\lambda^{-1} \circ S_1^{-1}(A) + \mu \circ \Pi_\lambda^{-1} \circ S_2^{-1}(A)).
\end{aligned}$$

Therefore $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$. □

2.3 Law of pure type

We will show that Bernoulli convolutions are of pure type.

Proposition 2.12. *ν_λ is either purely singular or absolutely continuous with respect to the Lebesgue measure $\mathcal{L}^1$.*

Proof. Throughout the whole paper, unless otherwise specified, absolute continuity and purely singularity are with respect to the Lebesgue measure $\mathcal{L}^1$.

By Lebesgue decomposition theorem, we can represent ν_λ as $\nu_{ac} + \nu_s$ where ν_{ac} and ν_s are absolutely continuous part and purely singular part with respect to $\mathcal{L}^1$ respectively. Notice that $S(\nu_{ac})$ remains to be absolutely continuous while $S(\nu_s)$ remains to be purely singular. Now

$$\nu_\lambda = S(\nu_\lambda) = S(\nu_{ac}) + S(\nu_s).$$

Since the Lebesgue decomposition is unique, ν_{ac} and ν_s are also invariants under S. By uniqueness of invariant probability measure, one of them have to vanish. $\square$

It is easily seen that ν_λ is purely singular for $\lambda < \frac{1}{2}$ as it is supported in a standard Cantor set. However, although we know ν_λ is of pure type for any λ, it is hard to determine whether ν_λ is purely singular or absolutely continuous for a particular given $\lambda \in (\frac{1}{2}, 1)$. Erdős [3] found the only known numbers where ν_λ is purely singular. They are the reciprocals of PV numbers. On the other hand, Garsia [6] found the largest known explicit set of numbers where ν_λ is absolutely continuous. The set consists of reciprocals of algebraic integers in $(1, 2)$ whose minimal polynomial has other roots outside the unit circle and the constant coefficient ± 2. For instance the polynomials $x^{n+p} - x^n - 2$ where $p, n \geq 1$ and $\max\{p, n\} \geq 2$ satisfies the criterion.

Here we will show a simple class of examples where ν_λ is absolutely continuous. These examples are discovered by Wintner [23].

Proposition 2.13. *For $\lambda = 2^{-\frac{1}{k}}$ where $k \in \mathbb{N}$, ν_λ is absolutely continuous.*

Proof. Notice that for $k = 1$, we have $\lambda = \frac{1}{2}$ and $\mu = \frac{1}{4}\mathcal{L}^1|_{[-2,2]}$ satisfies $\mu = S(\mu)$. Therefore $\nu_{\frac{1}{2}} = \mu = \frac{1}{4}\mathcal{L}^1|_{[-2,2]}$ is absolutely continuous.

For $k > 1$, $b_1 * b_{\lambda^k} * b_{\lambda^{2k}} * \ldots = \nu_{\frac{1}{2}}$. Therefore ν_λ is the convolution of $\nu_{\frac{1}{2}}$ and some other measure. Since $\nu_{\frac{1}{2}}$ is absolutely continuous, so do ν_λ. $\square$

Remark 2.14. Actually it is also easy to show that the density of ν_λ is in C^{k-2} for $\lambda = 2^{-\frac{1}{k}}$ with $k \geq 2$, by noticing

$$\nu_\lambda(\cdot) = \nu_{\frac{1}{2}}(\cdot) * \nu_{\frac{1}{2}}(\lambda \cdot) * \ldots * \nu_{\frac{1}{2}}(\lambda^{k-1} \cdot).$$

Chapter 3

Some results related to pure singularity

Determining for which value of λ ν_λ is purely singular is the major problem on Bernoulli convolutions. However up to now only the reciprocals of PV numbers are found. In this chapter, we will prove that when λ is the reciprocal of a PV number, ν_λ is purely singular. Then we will show that when λ is the reciprocal of a member of another class of algebraic integers, namely the Salem numbers, then ν_λ have some 'bad' behavior so that we believe reciprocals of Salem numbers are good candidates on finding singular Bernoulli convolutions. Finally we will introduce the weak separation condition which provides a way for us to check singularity and to analyze the behavior of Bernoulli convolutions associated with PV numbers.

3.1 The Pisot-Vijayaraghavan numbers

Erdős [3] found that when λ is the reciprocal of a Pisot-Vijayaraghavan number, ν_λ is purely singular. No numbers other than reciprocals of PV numbers have been found to satisfy this property.

Proposition 3.1. *(Erdős [3]) For $\lambda = \alpha^{-1}$ where $\alpha \in (1,2)$ is a PV number, ν_λ is purely singular.*

Proof. Recall that a PV number is a real algebraic integer greater than 1, and all its conjugates are of magnitude strictly less than 1. A characteristic of PV numbers is that $\|\alpha^n\| \to 0$ as n tends to infinity, where $\|x\|$ is the distance between x and its nearest integer. In fact, if $\alpha = \alpha_1$ is a PV number, m is the degree of its minimal polynomial and $\alpha_2, \alpha_3, \ldots, \alpha_m$ are the conjugates of α, then $\sum_{i=1}^m \alpha_i^n$ is always an integer for any integral $n \geq 0$. Therefore there exist $\theta < 1$ such that $\|\alpha^n\| = \|\sum_{i=2}^m \alpha_i^n\| < \theta^n$ for all $n \in \mathbb{N}$.

Suppose ν_λ is not purely singular, then it have to be absolutely continuous with respect to $\mathcal{L}^1$ since ν_λ is pure type. By Riemann-Lebesgue lemma, its Fourier transform $\widehat{\nu_\lambda}(\xi)$ necessarily tends to 0 as $\xi \to \infty$. We will show on the contrary that there exist a sequence ξ_n where $\widehat{\nu_\lambda}(\xi_n)$ is uniformly bounded away from zero.

$$|\widehat{\nu_\lambda}(\pi\alpha^k)| = \prod_{n=0}^{\infty} |\cos(\pi\alpha^k\lambda^n)| = \prod_{n=-k}^{\infty} |\cos(\pi\lambda^n)| \geq \prod_{n=-\infty}^{\infty} |\cos(\pi\lambda^n)|.$$

If $|\cos(\pi\lambda^n)| = 0$ for some $n \in \mathbb{Z}$, then $\alpha^{-n} = m+\frac{1}{2}$. So α satisfies $(2m-1)\alpha^n - 2 = 0$ and so do its conjugates. However, this contradicts to the assumption that α is a PV number in $(1,2)$.

Therefore none of the terms $\cos(\pi\lambda^n)$ vanishes. Choose $N \in \mathbb{N}$ such that $\pi\lambda^N < 1-\lambda$. Then

$$\begin{aligned}
\prod_{n=0}^{\infty} |\cos(\pi\lambda^n)| &= \prod_{n=0}^{N-1} |\cos(\pi\lambda^n)| \prod_{n=N}^{\infty} |\cos(\pi\lambda^n)| \\
&\geq \prod_{n=0}^{N-1} |\cos(\pi\lambda^n)| \prod_{n=N}^{\infty} |1-\pi\lambda^n| \\
&\geq \prod_{n=0}^{N-1} |\cos(\pi\lambda^n)| (1 - \sum_{n=N}^{\infty} \pi\lambda^n) \\
&= C_1 > 0.
\end{aligned}$$

On the other hand, let $N' \in \mathbb{N}$ be such that $\theta^{N'} < \frac{1}{2}$. Then

$$\begin{aligned}\prod_{n=-\infty}^{-N'} |\cos(\pi\lambda^n)| &= \prod_{n=N'}^{\infty} |\cos(\pi\alpha^n)| \\ &\geq \prod_{n=N'}^{\infty} |\cos(\pi\theta^n)| \\ &= C_2 > 0,\end{aligned}$$

where the last inequality holds using the same argument as above.

Therefore for any k, $|\widehat{\nu_\lambda}(\pi\alpha^k)| \geq C_1C_2\prod_{n=-N'+1}^{-1}|\cos(\pi\lambda^n)| = C_1C_2C_3 > 0$. Since C_i does not depend on k, ν_λ is not absolutely continuous. □

3.2 The Salem numbers

Although the simple method shown above have found all known $\lambda \in (\frac{1}{2}, 1)$ such that ν_λ is purely singular, no more examples can be found by checking whether the Fourier transform converges to 0. Salem [18] have found $\widehat{\nu_\lambda}(\xi) \to 0$ as $\xi \to \infty$ for all $\lambda \in (\frac{1}{2}, 1)$ except the reciprocals of PV numbers.

In fact, Kahane [11] noticed that for all $\lambda \in (0, 1)$ except a set of Hausdorff dimension 0, $\widehat{\nu_\lambda}$ decreases rapidly in the sense that there exist $\gamma > 0$ such that

$$\widehat{\nu_\lambda}(\xi) = O(\frac{1}{\xi^\gamma}).$$

It is then interesting to look for exceptions of rapid decreasing. The reciprocals of PV numbers obvious belong to the exceptions, and so do the reciprocals of Salem numbers.

Proposition 3.2. *(Peres, Schlag and Solomyak [20]) Let $\lambda = \theta^{-1}$ where θ is a Salem number. Then*

$$\limsup_{\xi\to\infty} |\widehat{\nu_\lambda}(\xi)||\xi|^\epsilon = \infty \quad \textit{for all } \epsilon > 0.$$

Proof. Recall that a Salem number is a real algebraic integer greater than 1, all its conjugates are of magnitude less than or equal to 1, and at least one of them is equal to 1. A characteristic of Salem number is that for any $\delta > 0$ there exists $t \geq 1$ such that

$$\|t\theta^n\| \leq \delta \quad \text{for all } n \geq 1. \tag{3.1}$$

Let $t > 1$ be chosen to satisfy the above criterion.

$$\begin{aligned}
|\widehat{\nu_\lambda}(\pi t\theta^k)| &= \prod_{n=-k}^{\infty} |\cos(\pi t\lambda^n)| \\
&= |\widehat{\nu_\lambda}(\pi t)| \prod_{n=1}^{k} |\cos(\pi t\theta^n)| \\
&\geq |\widehat{\nu_\lambda}(\pi t)| \prod_{n=1}^{k} |\cos(\pi\delta)| \quad \text{by (3.1)} \\
&\geq |\widehat{\nu_\lambda}(\pi t)|(1-2\delta)^k \\
&\geq |\widehat{\nu_\lambda}(\pi t)| c(\pi t\theta^k)^{-\epsilon},
\end{aligned}$$

where $\epsilon = -\frac{\log(1-2\delta)}{\log\theta}$. Since $\epsilon \to 0$ as $\delta \to 0$, it remains to show that $\widehat{\nu_\lambda}(\pi t) \neq 0$.

Suppose $\widehat{\nu_\lambda}(\pi t) = 0$, $t\lambda^n = m + \frac{1}{2}$ for some $n \geq 0$. But $\{\theta^k\}_{k\geq 0}$ is dense mod 1. Hence $\{t\theta^k\}_{k\geq 0}$ is dense mod $\frac{1}{2}$, which contradicts equation 3.1 for $\delta < \frac{1}{2}$. □

Since the rate of decrease in Fourier transform is often related to smoothness, the proposition suggests that Salem numbers are good candidates to be singular.

3.3 The weak separation condition

Lau and Ngai [15] introduced a condition on iterated function systems which includes those related to Bernoulli convolutions associated with PV numbers.

Definition 3.3. Let $S_i(x) = \rho x + a_i$ be contractions with $0 < \rho < 1$. Let $\Sigma_n = \{1, 2, \ldots, N\}^n$. For any $\sigma = \{\sigma_i\}_{i=0}^{n-1} \in \Sigma_n$, denote by $|\sigma| = n$ the length of σ, and by S_σ the function $S_{\sigma_0} \circ S_{\sigma_1} \circ \ldots \circ S_{\sigma_{n-1}}$.

We say that the *weak separation condition* holds on the iterated function system $\{S_i\}$ if there exist a constant $a > 0$ such that for any $|\sigma| = |\sigma'| = n \geq 1$,

$$S_\sigma(0) = S_{\sigma'}(0) \quad \text{or} \quad |S_\sigma(0) - S_{\sigma'}(0)| > a\rho^n.$$

Remark 3.4. The definition is originally introduced in a more general setting, where the space is $\mathbb{R}^n$ and contraction ratio may be different. Since the only iterated function system we are considering here is the simple one related to Bernoulli convolutions, we use this specific definition here for simplicity.

Proposition 3.5. *The iterated function system related to Bernoulli convolutions, $S_i = \lambda x + a_i$ for $1 \leq i \leq 2$ where $a_1 = 1$ and $a_2 = -1$, satisfies the weak separation condition when $\lambda = \alpha^{-1}$ is the reciprocal of a PV number.*

Proof. Notice that

$$\begin{aligned} |S_\sigma(0) - S_{\sigma'}(0)| &= |\sum_{i=0}^{n-1} a_{\sigma_{n-1-i}} \lambda^i - \sum_{i=0}^{n-1} a_{\sigma'_{n-1-i}} \lambda^i| \\ &= 2|\sum_{i=0}^{n-1} c_i \lambda^i| \quad \text{where } c_i \in \{-1, 0, 1\}. \end{aligned}$$

Therefore the weak separation condition holds if and only if there exist $a > 0$ such that for any $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} c_i \alpha^i = 0 \quad \text{or} \quad \left| \sum_{i=1}^{n} c_i \alpha^i \right| > a > 0.$$

The following method is taken from Garsia [7], Lemma 1.51. Let $\alpha_1 = \alpha$, $\{\alpha_i\}_{i=2}^m$ be the conjugates of α, and $P(x)$ be a polynomial with coefficients 0 or ± 1. Notice that $\prod_{i=1}^m P(\alpha_i)$ is a symmetric to α_i, therefore it needs to be an integer.

Suppose $P(\alpha) \neq 0$, so do $P(\alpha_i)$ for any $1 \leq i \leq m$. Then

$$
\begin{aligned}
|P(\alpha)| &= \left| \frac{\prod_{i=1}^m P(\alpha_i)}{\prod_{i=2}^m P(\alpha_i)} \right| \\
&\geq \left(\left| \prod_{i=2}^m P(\alpha_i) \right| \right)^{-1} \\
&\geq \prod_{i=2}^m \left(\sum_{j=0}^\infty |\alpha_i|^j \right)^{-1} \\
&= \prod_{i=2}^m (1 - |\alpha_i|) \quad \text{as } |\alpha_i| < 1 \\
&= a > 0.
\end{aligned}
$$

This completes the proof. □

There are several nice properties for the invariant measures related to the iterated function systems with weak separation condition. Lau, Ngai and Rao [16] discovered that the invariant measure μ is singular if one of the probability weight is greater than the contraction ratio. Also, μ is absolutely continuous only if its density is bounded.

Theorem 3.6. *(Lau, Ngai and Rao [16]) Suppose $\{S_i\}_{i=1}^N$ satisfies the weak separation condition, $p_i > 0$ and $\sum_{i=1}^N p_i = 1$. If $p_i > \rho$ for some i, then the invariant measure μ is purely singular.*

Before the proof of the theorem, let we introduce some notations and definitions.

Denote by $[\sigma]$ the equivalent class $\{\sigma' \in \Sigma_n \ : \ |\sigma'| = |\sigma| = n, S_\sigma = S_{\sigma'}\}$, by A_n the collection of equivalent classes $\{[\sigma] \ : \ \sigma \in \Sigma_n\}$, and by p_σ the product $\prod_{i=0}^{n-1} p_{\sigma_i}$. Let $I \ni 0$ be a non-empty compact interval satisfying

$$I \supseteq \bigcup_{i=1}^N S_i(I).$$

We will make use of the following lemmas. Intuitively, the first lemma tells that A_n cannot have too many members, while the second one tells that the proportion of space carrying a significant amount of mass cannot be too small.

Lemma 3.7. *Suppose $\{S_i\}_{i=1}^N$ satisfies the weak separation condition, then there exists $\gamma > 0$ such that $\#A_n \leq \gamma\rho^{-n}$.*

Proof of Lemma 3.7. Given that $|\sigma| = |\sigma'| = n$, $S_\sigma = S_{\sigma'}$ if and only if $S_\sigma(0) = S_{\sigma'}(0)$. Also notice that I satisfies $S_\sigma(0) \in I$ for any σ. By the definition of weak separation condition,

$$\#\{S_\sigma(0) \,:\, |\sigma| = n\} \leq \frac{\mathcal{L}^1(I)}{a\rho^n} + 1 \leq \gamma\rho^{-n}.$$

$\square$

Lemma 3.8. *Suppose $\{S_i\}_{i=1}^N$ satisfies the weak separation condition. For any $\Lambda \subseteq \{1, 2, \ldots, N\}^n$, let*

$$\tilde{\Lambda} = \left\{ \sigma \in \Lambda \,:\, \sum_{\sigma' \in [\sigma] \cap \Lambda} p_{\sigma'} > \frac{\rho^n}{4\gamma} \right\},$$

where γ is the constant in lemma 3.7. Then

$$\sum_{\sigma\in\Lambda} p_\sigma > \frac{1}{2} \quad \textit{implies} \sum_{\sigma\in\tilde{\Lambda}} p_\sigma > \frac{1}{4}.$$

Notice that $\sum_{\sigma'\in[\sigma]} p_{\sigma'}$ is actually a lower bound of $\mu(S_\sigma(I))$ because

$$\mu(A) = \sum_{\sigma\in\Sigma_n} p_\sigma \mu(S_\sigma^{-1}(A)). = \sum_{[\sigma]\in A_n} \left(\sum_{\sigma'\in[\sigma]} p_{\sigma'} \right) \mu(S_\sigma^{-1}(A))$$

Proof of Lemma 3.8.

$$\begin{aligned} \sum_{\sigma\in\Lambda\setminus\tilde{\Lambda}} p_\sigma &= \sum_{[\sigma]\in A_n} \sum_{\sigma'\in[\sigma]\cap(\Lambda\setminus\tilde{\Lambda})} p_{\sigma'} \\ &\leq \sum_{[\sigma]\in A_n} \frac{\rho^n}{4\gamma} \quad \text{by the definition of } \tilde{\Lambda} \\ &= \#A_n \frac{\rho^n}{4\gamma} \leq \frac{1}{4}. \end{aligned}$$

Hence

$$\sum_{\sigma \in \bar{\Lambda}} p_\sigma = \sum_{\sigma \in \Lambda} p_\sigma - \sum_{\sigma \in \Lambda \setminus \bar{\Lambda}} p_\sigma > \frac{1}{2} - \frac{1}{4} = \frac{1}{4}.$$

□

Proof of Theorem 3.6. We will show that for any $\epsilon > 0$ there exists $E_\epsilon \subseteq \mathbb{R}$ such that $\mu(E_\epsilon) \geq \frac{1}{2}$ while $\mathcal{L}^1(E_\epsilon) < \epsilon$. If that is the case,

$$E = \bigcap_{k \geq 1} \bigcup_{n \geq k} E_{2^{-n}}$$

satisfies $\mu(E) \geq \frac{1}{2}$ and $\mathcal{L}^1(E) = 0$, and so μ is purely singular.

Without loss of generality, we assume that $p_1 > \rho$, and $\gamma > \frac{1}{4}$. Let $r \in \mathbb{N}$ be such that

$$4\gamma \mathcal{L}^1(I) \left(\frac{\rho}{p_1}\right)^r < \epsilon.$$

We will show that there exists a sequence $\{\Lambda_i \in \Sigma_{ir}\}_{i=1}^s$ satisfying the following properties.

- They are disjoint. Here by disjoint we mean that for any $i < j$, $\sigma \in \Lambda_i$ and $\sigma' \in \Lambda_j$, we have $\sigma \neq \sigma'|_{ir}$, where the truncation $\sigma|_k$ is defined to be $\sigma_0\sigma_1 \ldots \sigma_{k-1} \in \Sigma_k$. Being disjoint ensures that when we want to expand μ by using $\mu = \sum_{i=1}^N p_i \mu \circ S_i^{-1}$ repeatedly, σ and σ' belongs to different terms.
- $\sum_{\sigma \in \Lambda_i} p_\sigma > \frac{p_1^r}{4}$. This shows that Λ_i have a significant amount of mass. Also,

$$\sum_{i=1}^{s} \sum_{\sigma \in \Lambda_i} p_\sigma \geq \frac{1}{2}.$$

- Let $E_i = \bigcup\{S_\sigma(I) \,:\, \sigma \in \Lambda_i\}$, then

$$\mathcal{L}^1(E_i) \leq \epsilon \sum_{\sigma \in \Lambda_i} p_\sigma \quad \text{for any } 1 \leq i \leq s.$$

The construction is simple using lemma 3.8.

1. Let $\Lambda_1 = (1, 1, \ldots, 1) \in \Sigma_r$.

2. Assume $\Lambda_1, \Lambda_2, \ldots, \Lambda_k$ have been constructed. If $\sum_{i=1}^{k} \sum_{\sigma \in \Lambda_i} p_\sigma \geq \frac{1}{2}$, terminate the algorithm with $s = k$.

3. Notice that now $\Lambda'_k = \{\sigma \in \Sigma_{kr} \,:\, \sigma|_{ir} \notin \Lambda_i, 1 \leq i \leq k\}$ satisfies $\sum_{\sigma \in \Lambda'_k} > \frac{1}{2}$, hence by lemma 3.8,
$$\tilde{\Lambda}_k = \left\{ \sigma \in \Lambda'_k \,:\, \sum_{\sigma' \in [\sigma] \cap \Lambda'_k} p_{\sigma'} > \frac{\rho^{kr}}{4\gamma} \right\} \quad \text{satisfies } \sum_{\sigma \in \tilde{\Lambda}_k} p_\sigma > \frac{1}{4}.$$

4. $$\Lambda_{k+1} = \{(\sigma, 1, 1, \ldots, 1) \in \Sigma_{(k+1)r} \,:\, \sigma \in \tilde{\Lambda}_k\}.$$

By construction it is obvious that Λ_i are disjoint and
$$\sum_{i=1}^{s} \sum_{\sigma \in \Lambda_i} p_\sigma \geq \frac{1}{2}.$$
When $i = 1$, $\sum_{\sigma \in \Lambda_i} p_\sigma = p_1^r \geq \frac{p_1^r}{4}$. When $i > 1$,
$$\sum_{\sigma \in \Lambda_i} p_\sigma = \sum_{\sigma \in \tilde{\Lambda}_{i-1}} p_\sigma p_1^r \geq \frac{p_1^r}{4}.$$
This also shows that the construction have to terminate in finite steps.

Finally, for any $\sigma \in \Lambda_i$,
$$\sum_{\sigma' \in [\sigma] \cap \Lambda_i} p_{\sigma'} > \frac{\rho^{(i-1)r}}{4\gamma} p_1^r = \frac{\rho^{ir}}{4\gamma} \left(\frac{p_1}{\rho}\right)^r > \frac{\rho^{ir}}{\epsilon} \mathcal{L}^1(I).$$
Therefore
$$\epsilon \sum_{\sigma \in \Lambda_i} p_\sigma > \epsilon \frac{\rho^{ir}}{\epsilon} \mathcal{L}^1(I) \#\{S_\sigma \,:\, \sigma \in \Lambda_i\} \geq \mathcal{L}^1(E_i).$$
Now let $E = \bigcup_{i=1}^{s} E_i$. Then
$$\mu(E) \geq \sum_{i=1}^{s} \sum_{\sigma \in \Lambda_i} p_\sigma > \frac{1}{2}.$$
Also
$$\mathcal{L}^1(E) \leq \sum_{i=1}^{s} \mathcal{L}^1(E_i) \leq \epsilon \sum_{i=1}^{s} \sum_{\sigma \in \Lambda_i} p_\sigma < \epsilon.$$
This complete the proof that μ is purely singular. □

Theorem 3.9. *(Lau, Ngai and Rao [16]) Suppose $\{S_i\}_{i=1}^N$ satisfies the weak separation condition. If the invariant measure μ is absolutely continuous, then its density function $f = D\mu$ is in $L^\infty(\mathbb{R})$.*

Proof. Suppose the density function $f \notin L^\infty(\mathbb{R})$.By Lebesgue density theorem, for any large M there exist a ball $B_{\rho^n}(x)$ such that

$$\frac{\mu(B_{\rho^n}(x))}{\rho^n} = \frac{1}{\rho^n}\int_{B_{\rho^n}(x)} f(t)dt > M. \tag{3.2}$$

Now view μ as another invariant measure by iterating $\{S_i\}_{i=1}^N$ n times.

$$\mu = \sum_{\sigma\in\Sigma_n} p_\sigma \mu \circ S_\sigma^{-1} = \sum_{[\sigma]\in A_n}\left(\sum_{\sigma\in[\sigma]} p_\sigma\right)\mu\circ S_\sigma^{-1}.$$

Put this in equation 3.2, we have

$$\sum_{[\sigma]\in A_n}\left(\sum_{\sigma\in[\sigma]} p_\sigma\right)\mu\circ S_\sigma^{-1}(B_{\rho^n}(x)) > M\rho^n.$$

Now $\mu(S_\sigma^{-1}(B_{\rho^n}(x))) \neq 0$ implies $S_\sigma(I)\cap B_{\rho^n}(x) \neq \emptyset$, which means that $S_\sigma(0) \in B_{\rho^n(1+\mathcal{L}^1(I))}(x)$. By the same argument in lemma 3.7, there exists constant $c > 0$ such that $\#\{[\sigma]\in A_n \,:\, S_\sigma(I)\cap B_{\rho^n}(x)\} < c$. Hence the left-hand side of equation 3.2 have at most c nonzero terms, and thus at least one of the coefficients $\sum_{\sigma\in[\sigma]} p_\sigma$ must be greater than $\frac{M}{c}\rho^n$. Choose $M > c$, by theorem 3.6, μ is singular, which contradicts the assumption. □

Although there is no simple way to find out for what λ the iterated function system related to Bernoulli convolution with parameter λ satisfies the weak separation condition, the weak separation condition provides an alternative way to check singularity of ν_λ. If λ is known to have satisfied the weak separation condition, singularity can be ensured by either iterating several times to find a probability weight greater than the contraction ratio, or to prove it has unbounded density.

Chapter 4

A proof of almost everywhere absolute continuity

In 1962, Garsia [6] conjectured that for a.e. $\lambda \in (\frac{1}{2}, 1)$, ν_λ is absolutely continuous. Solomyak [22](1995) proved this conjecture. Soon later Peres and Solomyak [21] presented a considerably simplified proof of the theorem. In this chapter, we will present this simplified proof.

The proof requires several lemmas. First of all, we will give a criterion of absolute continuity.

Definition 4.1. Let $B_r(x) = [x-r, x+r]$. The *lower derivative* of a measure μ is

$$\underline{D}(\mu, x) = \liminf_{r \searrow 0} (2r)^{-1} \mu(B_r(x)).$$

Lemma 4.2. *Let μ be a Radon measure on $\mathbb{R}$. μ is absolutely continuous if and only if $\underline{D}(\mu, x) < \infty$ for μ almost all $x \in \mathbb{R}$.*

Proof. Let us use the usual notation $\mu \ll \nu$ to denote μ is absolutely continuous with respect to ν.

Suppose $\mu \ll \mathcal{L}^1$, $\underline{D}(\mu, x)$ is a version of Radon-Nikodym derivative and is finite for $\mathcal{L}^1$ a.e. x. Since $\mu \ll \mathcal{L}^1$, it is also finite for μ a.e. x.

If $\underline{D}(\mu, x) \leq t$ for all $x \in A$, then $\mu(A) \leq t\mathcal{L}^1$. This result follows from Vitali covering theorem (see Mattila [17]). If $\mathcal{L}^1(A) = 0$, then

$$\begin{aligned}\mu(A) &= \mu(\{x \in A \,:\, \underline{D}(\mu, x) < \infty\}) \\ &= \lim_{t\to\infty} \mu(\{x \in A \,:\, \underline{D}(\mu, x) \leq t\}) \\ &\leq \lim_{t\to\infty} t\mathcal{L}^1(A) = 0.\end{aligned}$$

Therefore $\mu \ll \mathcal{L}^1$. □

The following lemma is used in the proof to estimate the measure of some particular sets.

Definition 4.3. Let $\mathcal{G}$ be the collection of all functions in the form of

$$g(x) = 1 + \sum_{n=1}^{\infty} c_n x^n \quad \text{where } c_n \in \{-1, 0, 1\}.$$

We say *δ-transversality condition* holds on an interval $I = [\lambda_0, \lambda_1] \subset [0, 1)$ if

$$\forall g \in \mathcal{G}, \forall x \in I, g(x) < \delta \Rightarrow g'(x) < -\delta.$$

This means that in the interval I, the graph of each $g \in \mathcal{G}$ crosses each horizontal lines below height δ transversally with slope at most $-\delta$.

Lemma 4.4. *Assume δ-transversality condition holds on* $I = [\lambda_0, \lambda_1] \subset [0, 1)$, *then for any* $\rho > 0$,

$$\mathcal{L}^1(\{\lambda \in I \,:\, |g(\lambda)| \leq \rho\}) \leq 2\delta^{-1}\rho.$$

Proof. This is obvious if $\rho \geq \delta$ as $\mathcal{L}^1(I) \leq 1$. For $\rho < \delta$, $|g(\lambda)| \leq \rho$ implies $g'(\lambda) < -\delta$. Notice that the set where $\{\lambda \in I \,:\, |g(\lambda)| \leq \rho < \delta\}$ is then an interval as g is strictly decreasing. As $|g'| > \delta$,

$$\mathcal{L}^1(\{\lambda \in I \,:\, |g(\lambda)| \leq \rho\}) \leq (2\rho)/(\delta) = 2\delta^{-1}\rho.$$

□

It seems hard to find and show an interval satisfying δ-transversality, in order to use the above estimation. However, the following lemma give a convenient way to find such an interval.

Definition 4.5. A function $h(x)$ is called a *(*)-function* if for some $k \geq 1$ and $a_k \in [-1, 1]$,

$$h(x) = 1 - \sum_{i=1}^{k-1} x^i + a_k x^k + \sum_{i=k+1}^{\infty} x_i.$$

Lemma 4.6. *Suppose that a (*)-function h satisfies $h(x_0) > \delta$ and $h'(x_0) < -\delta$ for some $x_0 \in (0,1)$ and $\delta \in (0,1)$. Then the transversality condition holds on $[0, x_0]$.*

Proof. First we will show that $h(x) > \delta$ and $h'(x) < -\delta$ for all $x \in [0, x_0]$.

Notice that h'' have at most one zero in $(0,1)$. Otherwise suppose $h''(x) = -\sum_{i=0}^{k-1} c_i x^i + \sum i = k^{\infty} c_i x^i$ where $c_i \geq 0$, and $x, y \in (0,1)$, $x < y$ are two zeros of it. Let $r = \frac{y}{x} > 1$.

$$\begin{aligned}\sum_{i=0}^{k-1} c_i y^i = \sum_{i=0}^{k-1} c_i r^i x^i \geq r^k \sum_{i=0}^{k-1} c_i x^i \\ = r^k \sum_{i=k}^{\infty} c_i x^i \geq \sum_{i=k}^{\infty} c_i r^i x^i = \sum_{i=k}^{\infty} c_i y^i.\end{aligned}$$

However, equality only holds when $c_i = 0$ for any $i \geq 0$, which leads to contradiction.

We have $h'(0) = -1 < -\delta$ if $k > 1$ and $h'(0) < h'(x_0) < -\delta$ otherwise. Thus $h'(0) < -\delta$. Since $\lim_{x \nearrow 1} h'(x) = +\infty$, we must have $h'(x) < -\delta$ for all $x \in (0, x_0)$, otherwise h'' would have at least two zeros. Now clearly, $h(x) > h(x_0) > \delta$ for $x \in (0, x_0)$.

Let $g \in \mathcal{G}$. Consider $f(x) = g(x) - h(x)$. Then $f(x) = \sum_{i=1}^{l} d_i x^i - \sum_{i=l+1}^{\infty} d_i x^i$, where $d_i \geq 0$ and $l = k-1$ or $l = k$. We have for any $x \in [0, x_0]$, by the claim proved above,

$$g(x) < \delta \Rightarrow f(x) < 0 \Rightarrow f'(x) < 0 \Rightarrow g'(x) < -\delta.$$

The middle implication is a consequence of one coefficient sign change.

$$\begin{aligned} f(x) < 0 &\Rightarrow l \sum_{i=1}^{l} c_i x^i < l \sum_{i=l+1}^{\infty} c_i x^i \\ &\Rightarrow \sum_{i=1}^{l} c_i i x^i < \sum_{i=l+1}^{\infty} c_i i x^i \\ &\Rightarrow \sum_{i=1}^{l} c_i i x^{i-1} < \sum_{i=l+1}^{\infty} c_i i x^{i-1} \\ &\Rightarrow f'(x) < 0. \end{aligned}$$

□

Theorem 4.7. *For $\mathcal{L}^1$ a.e. $\lambda \in (\frac{1}{2}, 1)$ the measure ν_λ is absolutely continuous and has an L^2-density.*

Proof. By Lemma 4.2, ν_λ is absolutely continuous if and only if $\underline{D}(\nu_\lambda, x) < \infty$ for ν_λ almost all $x \in \mathbb{R}$. If we can show that

$$S = \int_I \int_{\mathbb{R}} \underline{D}(\nu_\lambda, x) d\nu_\lambda(x) d\lambda < \infty,$$

then ν_λ is absolutely continuous for a.e. $\lambda \in I$.

Notice that for ν_λ is absolutely continuous, $\underline{D}(\nu_\lambda, x)$ is a version of the Radono-Nikodym derivative $\frac{d\nu_\lambda(x)}{dx}$, so $\int_{\mathbb{R}} \underline{D}(\nu_\lambda, x) d\nu_\lambda(x) < \infty$ actually implies that the density $\frac{d\nu_\lambda(x)}{dx} \in L^2(\mathbb{R})$.

By Fatou's Lemma,

$$S \le \liminf_{r \searrow 0} (2r)^{-1} \int_I \int_{\mathbb{R}} \nu_\lambda(B_r(x)) d\nu_\lambda(x) d\lambda. \tag{4.1}$$

Since $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$ and Π_λ is continuous, we can change variables in the equation (4.1).

$$S \le \liminf_{r \searrow 0} (2r)^{-1} \int_I \int_{\Omega} \nu_\lambda(B_r(\Pi_\lambda(\omega))) d\mu(\omega) d\lambda. \tag{4.2}$$

Next, denote by χ_A the characteristic function of a set A and use $\nu_\lambda = \mu \circ \Pi_\lambda^{-1}$ again. We have,

$$\nu_\lambda(B_r(\Pi_\lambda(\omega))) = \int_{\mathbb{R}} \chi_{B_r(\Pi_\lambda(\omega))} d\nu_\lambda = \int_{\Omega} \chi_{\{\tau \in \Omega \,:\, |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \le r\}} d\mu(\tau).$$

Substitute this into equation (4.2) and exchange the order of integration, we have

$$S \leq \liminf_{r \searrow 0} (2r)^{-1} \int_\Omega \int_\Omega \mathcal{L}^1\{\lambda \in I \, : \, |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \leq r\} d\mu(\tau) d\mu(\omega). \tag{4.3}$$

Let

$$\phi_{\tau,\omega}(\lambda) = \Pi_\lambda(\tau) - \Pi_\lambda(\omega) = \sum_{n=0}^{\infty} (\tau_n - \omega_n)\lambda^n.$$

Notice that $\tau_n - \omega_n \in \{-2, 0, 2\}$. We need to estimate $\mathcal{L}\{\lambda \in I \, : \, |\phi_{\tau,\omega}(\lambda)| \leq r\}$. We can write $\phi_{\tau,\omega}(\lambda) = 2\lambda^k g(\lambda)$, where $k = |\omega \wedge \tau| := \min\{n \, : \, \omega_n \neq \tau_n\}$ and g is a power series with constant term ± 1. Without loss of generality, $\omega_k < \tau_k$ so that $g \in \mathcal{G}$.

In order to estimate the integrand in equation (4.3), assume that δ-transversality condition holds on the interval $I = [\lambda_0, \lambda_1]$. $|\phi_{\tau,\omega}(\lambda)| \leq r$ implies that $|g(\lambda)| \leq \lambda_0^{-k} r/2$ for $\lambda \in I = [\lambda_0, \lambda_1]$. Applying Lemma 4.4 with $\rho = \lambda_0^{-k} r/2$, we obtain

$$\mathcal{L}^1\{\lambda \in I \, : \, |\phi_{\tau,\omega}(\lambda)| \leq r\} \leq \delta^{-1} \lambda_0^{-k} r.$$

Substituting this estimation in equation (4.3) yields

$$\begin{aligned} S &\leq \delta^{-1} \liminf_{r \searrow 0} (2r)^{-1} \int_\Omega \int_\Omega \lambda_0^{-|\omega \wedge \tau|} r \, d\mu(\tau) d\mu(\omega) \\ &= (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} (\mu \times \mu)(\{(\omega, \tau \, : \, |\omega \wedge \tau| = k\}) \qquad (4.4) \\ &= (2\delta)^{-1} \sum_{k=0}^{\infty} \lambda_0^{-k} 2^{-k-1} < \infty. \end{aligned}$$

The last step requires that $\lambda_0 > \frac{1}{2}$. This proves for a.e. $\lambda \in I$ ν_λ is absolutely continuous with L^2 density, assuming δ-transversality condition holds on I.

Notice that if ν_{λ^2} is absolutely continuous with L^2 density, then so do ν_λ. This is because $\nu_\lambda(\cdot) = \nu_{\lambda^2}(\cdot) * \nu_{\lambda^2}(\lambda \cdot)$. Therefore it suffices to show that ν_λ is purely singular for a.e. $\lambda \in (\frac{1}{2}, 2^{-1/2})$.

In order to apply Lemma 4.4 for δ-tranversality, it remains to find an appropriate (*)-function that satisfies the assumption in Lemma 4.6. Let

$$h(x) = 1 - x - x^2 - x^3 + 0.5x^4 + \sum_{i=5}^{\infty} x^i.$$

h is an appropriate (*)-function, satisfying $h(2^{-2/3}) > 0.07$ and $h'(2^{-2/3}) < -0.09$. So δ-transversality is verified for the interval $[0, 2^{-2/3}]$. This establishes absolute continuity for a.e. $\lambda \in [\frac{1}{2}, 2^{-2/3}]$.

However in this way we cannot cover $[2^{-2/3}, 2^{-1/2}]$ since there is a function in $\mathcal{G}$ with a double zero around $0.68 < 2^{-1/2}$ (see Solomyak [22]), and having a double zero obviously contradicts δ-transversality.

In order to cover this interval consider the "thinned" random series $Z_\lambda = \sum_{i \neq 2+3j} \pm\lambda^i$, with every third term removed. The distribution of Z_λ is

$$\mu_\lambda = b_1 * b_\lambda * b_{\lambda^3} * b_{\lambda^4} * \ldots$$

If μ_λ is absolutely continuous, then so is ν_λ as ν_λ is the convolution of μ_λ and some other measure. Absolute continuity of μ_λ can be shown using the same argument, except that the sequence space have to be replaced by $\Omega = \{-1, 0, 1\}^{\mathbb{N}}$ with measure $\tilde{\mu}$ according to which every third symbol is forced to be 0 and all other symbols are -1 or 1 with probability $\frac{1}{2}$.

Now notice that

$$(\tilde{\mu} \times \tilde{\mu})(\{(\omega, \tau) \ : \ |\omega \wedge \tau| = k\}) = 2^{\lfloor -\frac{2k}{3} \rfloor - 1}.$$

Therefore equation (4.4) still holds with μ replaced by $\tilde{\mu}$ when $\lambda_0 > \frac{2}{3}$.

Finally, δ-transversality is now needed only for a smaller class $\mathcal{G}' = \mathcal{G}_1 \cup \mathcal{G}_2$, where $\mathcal{G}_i$ consists of all function in the form of

$$g(x) = 1 + \sum_{n=1}^{\infty} c_n x^n \quad \text{where } c_n \in \{-1, 0, 1\}, c_{3j+i} = 0.$$

It is then natural to see that we can still applies Lemma 4.6 with (*)-function replaced by thinned (*)-function, which is a element in $\mathcal{G}_i$ satisfying there exist $k \in \mathbb{N}$ such that $c_n = -1$ for any $1 \leq n < k$ and $c_n = 1$ for any $n > k$.

Therefore it suffices to check that the following two thinned (*)-functions

$$h_1(x) = 1 - x^2 - x^3 - x^5 - x^6 + \sum_{j=3}^{\infty}(x^{3j-1} + x^{3j}) \in \mathcal{G}_1$$

and

$$h_2(x) = 1 - x - x^3 - x^4 + \sum_{j=2}^{\infty}(x^{3j} + x^{3j+1}) \in \mathcal{G}_2$$

satisfy $f(x) > \delta$ and $f'(x) < -\delta$ at $x = 2^{-1/2}$.

Thus δ-transversality holds on $[0, 2^{-1/2}]$ in the thinned class. μ_λ is then absolutely continuous for a.e. $\lambda \in (2^{-2/3}, 2^{-1/2})$ and so is ν_λ.

Combining with the previous result, ν_λ is absolutely continuous for a.e. $\lambda \in (\frac{1}{2}, 1)$, with L^2 density. □

Remark 4.8. Strictly speaking, one have to verify that $\int_I \int_{\mathbb{R}} \underline{D}(\nu_\lambda, x) d\nu_\lambda(x)$ is a $\mathcal{L}^1$ measurable function before applying the Fatou's lemma. This integral can be written as

$$\int_\Omega \liminf_{r \searrow 0} (2r)^{-1} \int_\Omega \chi_{\{\tau \in \Omega \,:\, |\Pi_\lambda(\tau) - \Pi_\lambda(\omega)| \le r\}} d\mu(\tau) d\mu(\omega).$$

and checking measurability of the last expression is routine.

Remark 4.9. Notice that the theorem immediately implies that for a.e. $\lambda \in (2^{-1/2}, 1)$, ν_λ has continuous density, because it is the convolution of two probability measures with L^2 density. It is not yet known whether this is true for a.e. $\lambda \in (\frac{1}{2}, 2^{-1/2})$.

Chapter 5

Other results and problems

In this chapter, we will state several other results about Bernoulli convolutions. The results are related to the entropy, dimensions, and some examples where Bernoulli convolutions behave badly. At the end, we will conclude our thesis by stating some open problems.

5.1 Entropy of Bernoulli convolutions

The investigation of entropy of ν_λ begins with Garsia [6], who considered the entropies of the finite partial convolutions ν_λ^n. Recall that the entropy of a discrete probability measure $\mu = \sum_{i=1}^N p_i \delta_{x_i}$ is defined as

$$H(\mu) = -\sum_{i=1}^N p_i \log p_i.$$

Clearly the finite partial convolutions ν_λ^n is discrete, so $H_N(\lambda) = H(\nu_\lambda^n)$ is well-defined. In fact, $H_N(\lambda)$ can be defined as in the following.

Let

$$d_N(\lambda) = \left\{ x \in \mathbb{R} \,:\, x = \sum_{n=0}^{N-1} a_n \lambda^n, a_n \in \{0,1\} \right\},$$

and for each $x \in d_N(\lambda)$, let

$$p_N(x) = 2^{-N} \# \left\{ (a_0, a_1, \ldots, a_{N-1}) \in \{0,1\}^N \,:\, x = \sum_{n=0}^{N-1} a_n \lambda^n \right\}.$$

Finally,

$$H_N(\lambda) = -\sum_{x \in d_N(\lambda)} p_n(x) \log p_n(x).$$

For simplicity we use $a_i \in \{0, 1\}$ here. It should be noticed that the values are not altered if $\{-1, 1\}$ is used instead.

For $\lambda \in (\frac{1}{2}, 1)$, Garsia considered the limit

$$G_\lambda = \lim_{N \to \infty} \frac{H_N(\lambda)}{N}.$$

Clearly when $\lambda \in (\frac{1}{2}, 1)$ does not satisfy an equation with coefficients $\{-1, 0, 1\}$, we have

$$H_N(\lambda) = N \log 2 \quad \text{and} \quad G_\lambda = \log 2,$$

otherwise $G_\lambda < \log 2$.

In [6], Garsia showed that the limit G_λ always exists for $\lambda \in (\frac{1}{2}, 1)$. Moreover when $G_\lambda < -\log \lambda$, ν_λ is purely singular, and this inequality holds when $\lambda \in (\frac{1}{2}, 1)$ is the reciprocal of a PV number. The value $H_\lambda = \frac{G_\lambda}{-\log \lambda}$ is named the *Garsia's entropy*.

Alexander and Zagier [1] analyzed the Fibonacci graph and estimated H_{λ_g} to high precision, where $\lambda_g = \frac{-1+\sqrt{5}}{2}$ is the reciprocal of the golden ratio. The method can be extended to Bernoulli convolutions associated with multinacci numbers τ_m, which are the positive root of $x^m + x^{m-1} + \ldots + x - 1 = 0$. It is known that $H_{\lambda_2} = H_{\lambda_g} \approx 0.9957$, $H_{\tau_3} \approx 0.9804$, and H_{τ_m} strictly increases to 1 for $m \geq 3$ exponentially fast.

Hare and Sidorov [8](2010) gave a loose global lower bound of Garsia's entropy for all Bernoulli convolutions associated with $\lambda \in (\frac{1}{2}, 1)$ which is the reciprocal of a PV number. They proved the following theorem.

Theorem 5.1. *(Hare and Sidorov [8]) Let $\lambda \in (\frac{1}{2}, 1)$ be the reciprocal of a PV number. Then we have $H_\lambda > 0.81$.*

We will explain their method.

Let $\lambda \in (\frac{1}{2}, 1)$ be the reciprocal of a PV number, define

$$E_n(x,\lambda) := \left\{ (a_0, a_1, \ldots, a_{n-1}) \in \{0,1\}^n \ :\ 0 \le x - \sum_{k=0}^{n-1} a_k \lambda^k \le \frac{1}{1-\lambda} \right\}.$$

Then $E_n(x, \lambda)$ is the set of all possible prefixes where x can be represented as $\sum_{k=0}^{\infty} a_k \lambda^k$.

The following lemma plays a central role in the method.

Lemma 5.2. *Suppose there exists $\alpha \in (1,2)$ such that $\#E_n(x,\lambda) = O(\alpha^n)$ for all $x \in [0, \frac{1}{1-\lambda}]$. Then*

$$H_\lambda \ge -\log_\lambda \frac{2}{\alpha}.$$

Proof of Lemma 5.2. Let $\{a_i\}_{i\ge 0}$ satisfies $x = \sum_{k=0}^{\infty} a_k \lambda^k$. Define

$$p_n(a_0, a_1, \ldots, a_{n-1}) = \# \left\{ (a'_0, a'_1, \ldots, a'_{n-1}) \in \{0,1\}^n \ :\ \sum_{k=0}^{n-1} a_k \lambda^k = \sum_{k=0}^{n-1} a'_k \lambda^k \right\}.$$

Then by Lalley [13], Theorems 1,2,

$$\sqrt[n]{p_n(a_0, a_1, \ldots, a_{n-1})} \to 2\lambda^{H_\lambda} \quad \text{for } \mathbb{P} - a.e. (a_0, a_1, \ldots) \in \{0,1\}^{\mathbb{N}}.$$

Since $p_n(a_0, a_1, \ldots, a_{n-1}) \le \#E_n(\sum_{k=0}^{n-1} a_k\lambda^k, \lambda)$, we have $\sqrt[n]{p_n(a_0, a_1, \ldots, a_{n-1})} \le \epsilon_n p$ with $\epsilon_n \to 1$.

Therefore $2\lambda^{H_\lambda} \le \alpha$ and the proof is completed. □

Sketch of proof of Theorem 5.1. Define the maximum growth exponent as

$$\mathcal{M}_\lambda := \sup_{x\in[0,\frac{1}{1-\lambda}]} \limsup_{n\to\infty} \sqrt[n]{\#E_n(x,\lambda)}.$$

Then by Lemma 5.2, $H_\lambda \ge -\log_\lambda \frac{2}{\mathcal{M}_\lambda}$.

For $r \ge 2$, let

$$R_r = \max_{x\in[0,\frac{1}{1-\lambda}]} \#E_r(x,\lambda).$$

Notice that $\#E_{n+r}(x,\lambda) \le R_r \#E_n(x,\lambda)$ for all $x \in [0, \frac{1}{1-\lambda}]$, $n \ge 1$ and $r \ge 2$. By induction this implies for any $r \ge 2$ and $x \in [0, \frac{1}{1-\lambda}]$, $E_n(x,\lambda) = O(R_r^{n/r})$. Therefore for any $r \ge 2$, $H_\lambda > -\log_\lambda \frac{2}{R_r^{1/r}}$.

Now we know how to give a lower bound for a particular $\lambda \in (\frac{1}{2}, 1)$. Observe that for fixed λ and $r \geq 2$, $\#E_r(x, \lambda)$ is a piecewise constant function over $x \in [0, \frac{1}{1-\lambda}]$. And R_r is also piecewise constant when it is considered as a function of $\lambda \in (\frac{1}{2}, 1)$. Therefore in order to compute for a global lower bound, we only need to check for the points λ where R_r jumps. The number of those transition points are finite and increases exponentially with r. The estimate 0.81 is obtained by considering $r = 14$. Though this method is simple, this method itself cannot give estimates with high precision. □

5.2 Dimensions

Apart from finding for which $\lambda \in (0, 1)$ ν_λ is purely singular, some authors investigates on dimensions of Bernoulli convolutions which are known to be singular, that is, those associated with reciprocals of PV numbers. Recall that the Hausdorff dimension of a measure μ on $\mathbb{R}$ is

$$\inf\{\dim E \ : \ E \subseteq \mathbb{R}, E \text{ Borel}, \mu(E) = \mu(\mathbb{R})\}.$$

Alexander and Yorke [2] proved that $H_\lambda = \frac{G_\lambda}{-\log \lambda}$ is always an upper bound of information dimension, and equality holds when λ is the reciprocal of a PV number. In fact, it is known that

$$\lambda^{-1} \text{ is Pisot } \Rightarrow \dim \nu_\lambda = H_\lambda.$$

Lau [14] introduced a new class of algebraic numbers, namely the F-numbers which include PV numbers, and calculated the exact mean-quadratic-variation dimension of Bernoulli convolutions associated with the F-numbers. ν_λ is proved to be purely singular associated with reciprocals of F-numbers. However, it is not known whether the set of F-numbers properly contains the set of PV numbers.

5.3 Non PV numbers with bad behavior

Solomyak [22] showed that for a.e. $\lambda \in (\frac{1}{2}, 1)$, ν_λ is absolutely continuous with L^2 density. A natural question arises as whether there exists $\lambda \in (\frac{1}{2}, 1)$ such that ν_λ is absolutely continuous, but without L^2 density.

Feng and Wang [5] gave a positive answer to the question. They studied several series of functions and discovered that there exist $\lambda \in (\frac{1}{2}, 1)$ where the density of ν_λ, if exists, is not in $L^2(\mathbb{R})$. An interesting fact is that the numbers they have found need not to be a PV number or a Salem number. They also found a large group of numbers where the density of ν_λ, if exists, is unbounded. The main results are stated as below.

Theorem 5.3. *(Feng and Wang [5]) Let $\lambda_{n,k}$ denote the reciprocal of the largest real root of the polynomial $P_{n,k}(x) = x^n - x^{n-1} - \ldots - x^k - 1$. For any $k \geq 3$ there exists $N(k) > 0$ such that for all $n \geq N(k)$ the density of $\nu_{\lambda_{n,k}}$, if exists, is not in $L^2(\mathbb{R})$. Moreover, $\lambda_{n,k}^{-1}$ is neither a PV number nor a Salem number for sufficiently large N.*

Theorem 5.4. *(Feng and Wang [5]) Let $\frac{1}{2} < \lambda < 1$ be a real root of a polynomial of degree n with all its coefficients -1, 0, or 1. Suppose $\lambda < 2^{-\frac{n}{n+1}}$, then the density of ν_λ, if exists, is unbounded.*

Remark 5.5. Notice that there are many numbers satisfying the condition. For example, for any polynomial with degree less than p and coefficients $-1, 0, 1$, the reciprocal of the largest real root of polynomial $x^n - x^{n-1} - \ldots - x^p + P(x)$ satisfies the condition for each sufficient large n.

5.4 Open problems

1. Suppose $\lambda \in (\frac{1}{2}, 1)$ and ν_λ is singular. Does that implies λ is the reciprocal of a PV number? This is the fundamental questions and no one have been

close to the solution.

2. Solomyak [22] found that ν_λ is continuous for a.e. $\lambda \in (2^{-1/2}, 1)$. Is it the same in the range $(\frac{1}{2}, 2^{-1/2})$?

3. Is it possible to find a non PV number such that the corresponding iterated function system satisfy the weak separation condition?

4. Can we find an efficient algorithm to estimate the Hausdorff dimension of a Bernoulli convolution up to arbitrary precision, when the associated parameter is not a PV number?

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